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Spacetime Virasoro algebra from strings on zero radius AdS_3

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Abstract: We study bosonic string theory in the light-cone gauge on AdS_3 spacetime with zero radius of curvature (in string units) $R/\sqrt{\alpha'} = 0$. We find that the worldsheet theory admits an infinite number of conserved quantities which are naturally interpreted as spacetime charges and which form a representation of (two commuting copies of) a Virasoro algebra. Near the boundary of AdS_3 these charges are found to be isomorphic to the infinite set of asymptotic Killing vectors of AdS_3 found originally by Brown and Henneaux. In addition to the spacetime Virasoro algebra, there is a worldsheet Virasoro algebra that generates diffeomorphisms of the spatial coordinate of the string worldsheet. We find that if the worldsheet Virasoro algebra has a central extension then the spacetime Virasoro algebra acquires a central extension via a mechanism similar to that encountered in the context of the $\mathrm{SL}(2,\mathbb{R})$ WZW model. Our observations are consistent with a recently proposed duality between bosonic strings on zero radius AdS_{d+1} and free field theory in d dimensions.

Keywords: Bosonic Strings, AdS-CFT and dS-CFT Correspondence, Space-Time Symmetries.

Contents

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1. Introduction

The purpose of this paper is to study certain aspects of string theory on AdS_3 spacetime with zero radius of curvature (in string units). In certain contexts, string theories on infinitely curved AdS spaces have been conjectured to be dual to free field theories [1]–[4]. Such a duality is expected to occur as a limiting case (of zero 't Hooft coupling) in the classic example of the holographic correspondence between type-IIB superstring theory on $AdS_5 \times S^5$ and the SU(N) $\mathcal{N}=4$ supersymmetric Yang-Mills theory in four-dimensional Minkowski space [6, 7].[1] Some evidence in support of this expectation has been presented in [3, 5].

[1]There the 't Hooft coupling λ of the gauge theory is related to the radius of curvature R of the AdS background via $R^2 = \alpha' \sqrt{\lambda}$ whilst $1/N$ controls the string genus expansion. Pushing Maldacena's conjecture to the limit, we would then conclude that at $\lambda = 0$ (and $N \to \infty$) the IIB theory on $AdS_5 \times S^5$ with vanishing radius should be dual to the free $\mathcal{N}=4$ theory.

It has been further argued in [1, 2] that (noninteracting) bosonic closed string theory on AdS_{d+1} with zero radius of curvature is dual to the free scalar field theory of $N \times N$ matrices (with $N \to \infty$) in d spacetime dimensions. In particular, it was shown that the hamiltonian and states of the string theory in light-cone gauge at $R^2/\alpha' = 0$ naturally map to the corresponding objects in the sector of single-trace states of the free matrix-valued scalar field theory. Unlike the case of the $\mathcal{N} = 4$ theory it is not clear if this correspondence can be extended to an interacting field theory and this appears to be related to the fact that we cannot treat R^2/α' as a small non-zero parameter in the string theory. Nevertheless, the proposal of [1, 2] raises certain intriguing questions — even at the level of the free theory. One such question that forms the motivation for this note occurs in the $d = 2$ case wherein we would expect bosonic string theory on AdS_3 in the singular limit to be dual to the free boson theory in two dimensions. Free field theory in two dimensions is special in that it has an infinite-dimensional conformal symmetry generated by two copies of the Virasoro algebra with a central extension. The question is whether bosonic closed string theory on zero radius AdS_3 can also realise (two commuting copies of) a *spacetime* Virasoro algebra with a non-zero central extension, which could then potentially be identified as the conformal algebra of a holographic dual field theory. We find that this is indeed the case.

This question can also be posed in the context of type-IIB strings on $AdS_3 \times S^3 \times T^4$ with R-R three-form background, which is the near horizon limit of the D1-D5 brane system. In the limit of zero AdS radius ($R/\sqrt{\alpha'} \to 0$) one expects that this theory is likely dual to the orbifold point of the CFT of the symmetric product space $(T^4)^k/S_k$ (where $k = Q_1 Q_5$ for Q_1 D1-branes and Q_5 D5-branes). The light-cone superstring hamiltonian in this background has been derived in [8, 9]. The analysis in our paper applies to the bosonic sector of this theory in the tensionless limit.[2]

What makes our analysis possible is the well-known fact that, in light-cone gauge, the classical string action on AdS space (in Poincaré coordinates) with $R^2/\alpha' = 0$ undergoes a drastic simplification [12, 1, 2, 3]. In particular, spatial gradient terms on the worldsheet drop out and the string breaks up into noninteracting bits. In this sense the string indeed appears to be tensionless, which agrees with a naive interpretation of the $R^2/\alpha' \to 0$ limit as taking α' (the inverse tension) to infinity whilst keeping R fixed. We are able to show that this theory satisfies the basic requirement for consistency, namely that the spacetime isometry algebra of AdS_3, realised on the light-cone worldsheet, closes in the quantum theory.

One of our main observations is that the tensionless string on AdS_3 admits an infinite set of exactly conserved quantities (constructed from worldsheet fields) which satisfy a Virasoro algebra and which can naturally be interpreted as spacetime charges. The interpretation of these conserved quantities as spacetime charges follows from two facts. First, the $\mathrm{SO}(2,2) \cong \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ isometry group of AdS_3 is a global symmetry of the worldsheet theory and the corresponding Noether charges are in fact a subset of

[2] The interpretation of the tensionless limit in the S-dual setting, namely the F1-NS5 brane system is not clear to us. In the context of the Wess-Zumino-Witten (WZW) model description of the theory, this limit has been discussed recently in [10].

the infinite set of conserved quantities we consider. Second, in an asymptotic sense, these conserved quantities are isomorphic to the Brown-Henneaux Killing vectors which generate the asymptotic isometry group of AdS_3 [7, 13]. It should be pointed out that the Brown-Henneaux isometries are not symmetries of the string action. In fact we find that our conserved quantities can be interpreted as the Noether charges associated with certain global symmetries of the string action in light-cone gauge. These symmetries are generated by global (nonlinear) field redefinitions which match up with the Brown-Henneaux transformations only in an asymptotic sense.

Furthermore we find in the first quantised string theory a mechanism for inducing a spacetime central charge which is similar to the mechanism encountered in the $\mathrm{SL}(2,\mathbb{R})$ WZW model in [14]. In particular the worldsheet theory of the closed tensionless string in light-cone gauge has a residual invariance generated by diffeomorphisms of the circle (the spatial coordinate of the closed string worldsheet), whose generators form a *worldsheet* Virasoro algebra. We find that if this worldsheet algebra is allowed to have a central extension, it naturally induces a central term in the spacetime algebra due to the 'winding' of certain time-independent closed string fields. A very similar mechanism was found in [14] in the context of the $\mathrm{SL}(2,\mathbb{R})$ WZW model wherein the spacetime algebra can be constructed from the affine algebra on the worldsheet and a non-zero spacetime central extension appears due to winding of long string states in AdS_3. The existence of a spacetime Virasoro algebra is encouraging as it is in accord with the expectation that a theory of gravity on AdS_3 should (at least asymptotically) have an enhanced symmetry generated by two commuting Virasoro algebras. The observations above are consistent with the expectation that the tensionless string on AdS_3 should be dual to a conformal field theory in two dimensions.

The outline of our paper is as follows. In section 2 we briefly review the classical light-cone gauge action for strings on AdS space and the limit of zero AdS radius. We also discuss canonical quantisation of the zero radius system and the presence of a worldsheet Virasoro algebra. In section 3 we construct a set of conserved quantities on the worldsheet of the tensionless string and we show that they are to be identified with the generators of a spacetime Virasoro algebra. In section 4, we present our conclusions and discussions. Finally, in the appendix we study issues pertaining to operator ordering and the closure of the algebra in the quantum theory. We also point out the existence of (worldsheet realisations of) spacetime Virasoro algebras for AdS spaces of arbitrary dimension.

2. Bosonic light-cone strings on AdS

2.1 The classical theory

In this section we briefly review some aspects of bosonic strings on AdS space in the light-cone gauge. Detailed discussions of this topic can be found in [2, 11, 8].[3] We begin

[3] A covariant formulation of tensionless strings on AdS space has been discussed recently in [15]. The tensionless string in curved space and perturbations about that limit have also been studied in detail in [16]

by considering classical string propagation on the patch of AdS_3 covered by the Poincaré coordinates (x, t, z). In these coordinates, the AdS_3 metric is

$$ds^2 = \frac{R^2}{z^2}(2dx^+dx^- + dz^2)\,, \tag{2.1}$$

where x^+ and x^- are the light-cone variables defined by $x^\pm := (x \pm t)/\sqrt{2}$. The radial coordinate z is non-negative and $z = 0$ corresponds to the boundary of AdS_3. On the string worldsheet, parametrised by the coordinates (τ, σ), we can make the standard light-cone gauge choice

$$x^+ \equiv \frac{x+t}{\sqrt{2}} = p_-\tau\,, \tag{2.2}$$

where the constant p_- will subsequently be identified with the light-cone momentum. Unlike in flat space, we do not have the freedom to choose a conformal gauge for the worldsheet metric h_{ij}. Instead we can only choose worldsheet reparametrisations to set $h_{01} = 0$. With these gauge choices, the string action is

$$S_{LC} = \int d^2\sigma \frac{R^2}{4\pi\alpha' z^2}\left[-\sqrt{\frac{-h_{11}}{h_{00}}}(2p_-\,\dot{x}^- + \dot{z}^2) + \sqrt{\frac{-h_{00}}{h_{11}}}(z')^2\right], \tag{2.3}$$

where $\dot{z} := \partial_\tau z$ and $z' := \partial_\sigma z$. The canonical momentum conjugate to x^- is $\pi_-(\sigma) = -p_-\sqrt{-h_{11}/h_{00}}\,R^2/2\pi\alpha' z^2$ and is a function of σ only. Residual reparametrisations of the form $\sigma \to \tilde{\sigma}(\sigma)$ can then be used to set $\pi_-(\sigma) = p_-$. This corresponds to a uniform distribution of the light-cone momentum on the worldsheet. The light-cone string action on AdS_3 is then[4]

$$S_{LC} = \int d\tau \int_0^1 d\sigma\left[p_-\,\dot{x}^- + \frac{1}{2}\dot{z}^2 - \frac{1}{8\pi^2\alpha'^2}\frac{R^4}{z^4}z'^2\right]. \tag{2.4}$$

As pointed out in [2], when compared with the light-cone action for strings in a flat background, eq. (2.4) (or the corresponding hamiltonian) can naturally be interpreted as the action for a string with variable tension $R^2/z^2(2\pi\alpha')$. With this form of the action we can now freely set the dimensionless parameter $R^2/\alpha' = 0$ and treat it as our definition of the classical light-cone string action on zero radius AdS_3 or equivalently bosonic tensionless strings on AdS_3:

$$S_{LC} = \int d\tau \int_0^1 d\sigma\left[p_-\,\dot{x}^- + \frac{1}{2}\dot{z}^2\right]; \qquad \frac{R^2}{\alpha'} = 0\,. \tag{2.5}$$

Note that all of the above can be simply extended to general AdS_d backgrounds by replacing z with z^i $(i = 1, \ldots, d-2)$ the $d-2$ transverse coordinates. For the sake of clarity however, we have restricted ourselves to AdS_3 as this is the subject of the note. The light-cone action must, of course, be supplemented with two Virasoro constraints that follow by varying the action with respect to the two independent components of the worldsheet metric. As usual,

[4] We remark that our convention differs slightly from that used in [2, 8]. The latter would have led to an action $\int p_-\dot{z}^2/2$ for the transverse mode. However, this does not alter any of the conclusions in this note.

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these constrain $x^-(\sigma,\tau)$ such that, except for its zero mode, it is completely determined in terms of the transverse degrees of freedom (in this case $z(\sigma,\tau)$). In the singular limit, the constraints are

$$p_-\,\dot{x}^- + \frac{1}{2}\,\dot{z}^2 = 0\,, \tag{2.6}$$

$$p_-\,x^{-\prime} + \dot{z}\,z' = 0\,. \tag{2.7}$$

In this limit, the above light-cone action eq. (2.5) is remarkably simple. Since the σ-derivatives drop out the string breaks up into free 'bits' and the action for the field $z(\sigma,\tau)$ simply describes an infinite number of free non-relativistic particles. However, the stringiness of the system is inherent in the constraints above. It is also clear from eq. (2.4) why a small non-zero value for R^2/α' cannot obviously be treated as a perturbation — the terms involving this small parameter become singular near the boundary $z \to 0$. Therefore the interpretation of the theory is clear only at zero AdS radius. Although the latter leads to a trivial action eq. (2.5), when supplemented with the constraints eqs. (2.6) and (2.7), we will see that it exhibits certain features expected in a theory of gravity on AdS_3. It is also interesting to note that the light-cone action for the tensionless string on AdS spaces coincides precisely with the so-called null string on flat space (see e.g. [17, 18]).

2.2 Canonical quantization

From the action eq. (2.5) and the constraints eqs. (2.6) and (2.7), it follows that both the fields $z(\sigma,\tau)$ and $x^-(\sigma,\tau)$ satisfy the free particle equations

$$\ddot{z}(\sigma,\tau) = 0\,, \qquad \ddot{x}^-(\sigma,\tau) = 0 \tag{2.8}$$

and therefore grow linearly with time. In order to perform canonical quantization it is convenient to expand the σ-dependence of the worldsheet fields in terms of their Fourier modes on the closed string:

$$z(\sigma,\tau) = \sum_{n=-\infty}^{\infty} (z_n + p_n\tau)e^{2\pi i n\sigma}\,; \qquad z_{-n} = z_n^\dagger,\, p_{-n} = p_n^\dagger\,, \tag{2.9}$$

which is real and periodic. Periodicity of z, however, does not guarantee the periodicity of the constrained variable x^- on the closed string. Indeed the general solution of the constraints eqs. (2.6) and (2.7) and the equations of motion (2.8) leads to the mode decomposition

$$x^-(\sigma,\tau) = \sum_{n=-\infty}^{\infty} (x_n^- + p_n^-\tau)e^{2\pi i n\sigma} + P_\sigma\,\sigma\,, \tag{2.10}$$

where

$$x_n^-(n\neq 0) = \sum_{m=-\infty}^{\infty} \frac{1}{p_-}\left(\frac{m}{n}-1\right) z_{n-m}p_m\,, \qquad P_\sigma = 2\pi i \sum_{n=-\infty}^{\infty} \frac{1}{p_-} n\, z_{-n}p_n\,. \tag{2.11}$$

The formula for p_n^- can also be deduced from the constraint eq. (2.6) (see e.g. [2]). Clearly the coefficient of the linear term P_σ does not automatically come out to be zero. It is the momentum flow in the σ-direction on the worldsheet. In fact, in the quantum theory, the vanishing of P_σ must be treated as a constraint to be imposed on physical states of the closed string, such that

$$P_\sigma|\mathrm{phys}\rangle = 0\,, \tag{2.12}$$

which is equivalent to the condition that $(x^-(1) - x^-(0))$ annihilates physical states. In the context of the ordinary tensionful string in flat space the analogous requirement leads to the level matching condition.

With the exception of its zero mode x_0^-, the field x^- is completely determined in terms of the radial variable z of AdS_3, via the constraint equations (2.6) and (2.7). Therefore x_0^- and $z(\sigma,\tau)$ are the two dynamical variables satisfying the canonical (equal time) commutation relations

$$[x_0^-, p_-] = i\,, \qquad [z(\sigma,\tau), \dot{z}(\sigma',\tau)] = i\delta(\sigma - \sigma')\,. \tag{2.13}$$

The second commutator implies that the Fourier modes of z satisfy

$$[z_m, p_n] = i\delta_{m+n,0}\,, \tag{2.14}$$

with all other commutators vanishing.[5] The canonical momenta conjugate to the coordinates x^+ and z are

$$\pi_+ = \dot{x}^- = -\frac{1}{2p_-}\dot{z}^2\,, \qquad \pi_z = \dot{z}\,. \tag{2.15}$$

The light-cone action and hamiltonian for the transverse (radial) coordinate are respectively,

$$S_{LC} = \int d^2\sigma\,\frac{1}{2}\dot{z}^2\,, \qquad H_{LC} = \frac{1}{2}\sum_{n=-\infty}^{\infty} p_n p_{-n} \tag{2.16}$$

which, of course, describe non-relativistic free particles.

The canonical commutation relations above must be appended with a consistent definition of quantum operators. This is clear in the case of the operator P_σ in eq. (2.11) which has an ordering ambiguity. In ordinary tensionful string theory where the worldsheet is a theory of oscillators there is a natural ordering prescription for quantum operators, namely normal ordering. Here however, the worldsheet theory is not a theory of oscillators and we need to consider the possible consistent definitions of the quantum operators. The light-cone hamiltonian eq. (2.16) has no ordering ambiguities. We discuss some of these ordering issues in appendix A.

[5] Our expressions for x_n^- and the canonical commutators differ from those in [2, 3] by a factor of $1/p_-$. This can be traced back to the normalisation of the action. The two conventions are related by a p_--dependent rescaling of the modes z_n and p_n. All final results are independent of choice of convention.

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2.3 Worldsheet Virasoro algebra

Interestingly, as in the case of ordinary tensile string theory, in the tensionless theory the Fourier modes y_n of the field $p_- x^{-\prime}$ actually constitute a Virasoro algebra,

$$[y_m, y_n] = (m-n)\, y_{m+n} + (cm^3 + c'm)\, \delta_{m+n,0} \tag{2.17}$$

where

$$y_n = -i \sum_{m=-\infty}^{\infty} (n-m) z_{n-m} p_m \,. \tag{2.18}$$

Note that we have allowed for the most general central extension to the worldsheet algebra. The values of c and c' depend delicately on the ordering of quantum operators. It has been observed in earlier works, in the context of tensionless strings in flat space (null strings) [17], that depending on the ordering prescriptions (and the related definition of the vacuum) c and c' can be non-zero. We review some of these details in appendix A. In what follows we assume that c has some non-zero value and $c' = 0$ and observe the consequences of this assumption. This choice is motivated by the ordering prescriptions introduced in [17] which we discuss in appendix A.[6]

At this point we find it convenient to introduce a new worldsheet variable $X(\sigma)$, defined as

$$X(\sigma) \ := \ x^-(\sigma,\tau) + \frac{1}{2p_-} z\dot{z}(\sigma,\tau)\,. \tag{2.19}$$

The main property of this field $X(\sigma)$ is that it is conserved by virtue of the constraints eqs. (2.6) and (2.7). $X(\sigma)$ is time-independent at every point on the string and encodes an infinite number of conserved quantities on the worldsheet. In addition, the spatial derivative of this field satisfies

$$p_- X' = \frac{1}{2}(\dot{z}' z - \dot{z} z') \tag{2.20}$$

and so its Fourier modes l_n are found to also satisfy a Virasoro algebra

$$[l_m, l_n] = (m-n)\, l_{m+n} - \frac{c}{2} m^3 \delta_{m+n,0}\,, \tag{2.21}$$

where

$$l_n = -\frac{i}{2} \sum_{m=-\infty}^{\infty} (n-2m) z_{n-m} p_m \tag{2.22}$$

and c has the same value as in eq. (2.17). One can understand the appearance of this Virasoro algebra as a consequence of a residual invariance of the gauge-fixed action generated by reparametrisations of the circle. In fact the light-cone action eq. (2.16) is invariant under the infinitesimal transformations $\delta z = z' f(\sigma) + z f'(\sigma)/2$ which can be thought of as a combination of a reparametrisation $\delta\sigma = f(\sigma)$ and a 'spacetime' conformal transformation. $\{l_n\}$ are the conserved Noether charges associated with this invariance and satisfy the algebra generated by infinitesimal diffeomorphisms of the circle.

[6] Following a normal ordering prescription it also turns out that $c = 2$. Note that c' can be set to zero by a suitable choice of an ordering constant in y_0 (which is equal to $p_- P_\sigma / 2\pi$).

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With all the above properties to hand, a straightforward calculation shows that $X(\sigma)$ obeys the following commutation relation

$$[X(\sigma), X(\sigma')] = \frac{i}{2p_-}\left(X'(\sigma) + X'(\sigma')\right)\left(\theta(\sigma - \sigma') - \theta(\sigma' - \sigma)\right) + \\ + \frac{i}{p_-}\left(\sigma' X'(\sigma') - \sigma X'(\sigma)\right) - \frac{1}{2\pi i}\frac{c}{2p_-^2}\delta'(\sigma - \sigma'). \qquad (2.23)$$

The time-independent field $X(\sigma)$ and its commutators form the basis for our subsequent spacetime interpretation of the worldsheet theory at zero AdS radius.

3. A spacetime Virasoro algebra

On general grounds [13], a theory of gravity on (asymptotically) AdS_3 spacetimes is expected to have an infinite-dimensional symmetry group generated by two commuting copies of a Virasoro algebra, thus leading to an interpretation as a spacetime conformal field theory in two dimesions. This holographic interpretation arises from the fact that although gravity in three dimensions does not have any propagating degrees of freedom, it has so-called global degrees of freedom corresponding to the degrees of freedom of the graviton that can be gauge transformed to the AdS_3 boundary. In the context of string theory on $AdS_3 \times S^3 \times M^4$ with non-zero NS-NS 2-form flux (the $\mathrm{SL}(2,\mathbb{R})$ WZW model), this phenomenon has been understood in a series of beautiful papers [14, 19, 20].

For tensionless strings on AdS space, in the absence of an understanding of the complete theory we cannot conclude that similar spacetime interpretations will apply. However, in light of the proposal of [1, 2] which suggests a holographic relation between bosonic strings on zero radius AdS_{d+1} and free field theory in d dimensions, we expect that a spacetime CFT could emerge for these strings on AdS_3. For such a description to exist we must be able to construct a candidate spacetime conformal algebra with a non-zero central charge (the latter being a necessary requirement for a holographic interpretation as a unitary conformal field theory in two dimensions). Below we show that tensionless strings on AdS_3 seem to naturally provide such a structure.

We propose that the generators of one copy of the spacetime Virasoro algebra are the time-independent quantities on the worldsheet given by

$$L_n := -i\int_0^1 d\sigma\, p_- X^{n+1} = -i\int_0^1 d\sigma\, p_-\left[x^- + \frac{z\dot{z}}{2p_-}\right]^{n+1}, \qquad (3.1)$$

for all integers n (for $n < -1$ we need to be careful in defining L_n as $X(\sigma)$ can have zeros). Strictly speaking these are the classical expressions. In the quantum theory we need to adopt an ordering prescription to define these operators (we will address this in appendices A and B). We first note certain important properties of these objects. By construction, they are time-independent and commute with the light-cone string hamiltonian since $X(\sigma)$ (eq. (2.19)) is time-independent as a consequence of the constraints.

Using the canonical commutators, it can be easily checked that the three elements

$$L_1 = -i \int_0^1 d\sigma \left[p_-(x^-)^2 + x^- z\dot{z} + \frac{1}{4p_-} z^2 \dot{z}^2 \right], \tag{3.2a}$$

$$L_0 = -i \int_0^1 d\sigma \left[p_- x^- + \frac{1}{2} z\dot{z} \right], \tag{3.2b}$$

$$L_{-1} = -ip_- , \tag{3.2c}$$

generate an $sl(2,\mathbb{R})$ algebra. Again, these are classical expressions and are subject to a specific ordering in the quantum theory. At the classical level, closure of the $sl(2,\mathbb{R})$ algebra follows from canonical Poisson bracket relations. As we discuss in appendix B, it is straightforward to check closure on appropriately defined physical states (defined via eq. (2.12)) in the quantum case as well. The $\mathrm{SO}(2,2) \cong \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ isometry group of AdS_3 acts as a global symmetry on worldsheet fields. L_1, L_0 and L_{-1} are the conserved charges associated with one of these two $\mathrm{SL}(2,\mathbb{R})$ invariances.

That $\{L_n\}$ should actually be thought of as spacetime charges in AdS_3 emerges when we expand out eq. (3.1) as a binomial series (we may think of this as a small z expansion), to obtain

$$L_n = -i \int_0^1 d\sigma \left[p_- \,(x^-)^{n+1} + \frac{1}{2}(n+1)(x^-)^n z\dot{z} + \frac{1}{8p_-} n(n+1)(x^-)^{n-1} z^2 \dot{z}^2 + \cdots \right]. \tag{3.3}$$

Recall that p_-, $\pi_+ = \dot{x}^- = -\dot{z}^2/2p_-$ and $\pi_z = \dot{z}$ are the canonical momenta conjugate to the coordinates x^-, x^+ and z respectively. Hence the leading terms in an expansion near the AdS_3 boundary become

$$L_n = -i \int_0^1 d\sigma \left[p_- \,(x^-)^{n+1} + \frac{1}{2}(n+1)(x^-)^n \, z\, \pi_z - \frac{1}{4} n(n+1)(x^-)^{n-1} \, z^2 \, \pi_+ + \cdots \right]. \tag{3.4}$$

In Poincaré coordinates, these are precisely the asymptotic isometry generators of AdS_3, realised on string worldsheet fields. Recall that one set of asymptotic Killing vectors of AdS_3 are

$$\begin{aligned} \xi_n^+ &= -\frac{1}{4} n(n+1)(x^-)^{n-1} z^2 + \mathcal{O}(z^4), \\ \xi_n^- &= (x^-)^{n+1} + \mathcal{O}(z^4), \\ \xi_n^z &= \frac{1}{2}(n+1)(x^-)^n z + \mathcal{O}(z^3), \end{aligned} \tag{3.5}$$

for all integers n (in Poincaré coordinates). It is easily seen that these vectors satisfy a Virasoro algebra while the central term in the algebra was otained after a very careful analysis in [13]. It is interesting that the infinite set of time-independent quantities $\{L_n\}$, on the worldsheet theory of the tensionless string, naturally have a geometric interpretation (in the $z \to 0$ limit) in terms of the asymptotic isometry generators of AdS_3. This relation to the Brown-Henneaux Killing vectors applies only to the first three terms in the binomial expansion of eq. (3.1) given in eq. (3.4). It is not clear to us whether the remaining terms have a natural geometrical interpretation. In light of the above, it is natural to think of $\{L_n\}$ as spacetime charges.

Although we have established a link between the worldsheet $\{L_n\}$ and the Brown-Henneaux generators, it is by no means obvious that the former constitute a Virasoro algebra. However, this fact follows from the highly nontrivial commutation relation obeyed by the worldsheet field $X(\sigma)$. This is relatively easy to establish at the classical level via Poisson brackets, since

$$\begin{aligned}[][L_m, L_n]_{PB} = -\int_0^1 d\sigma \int_0^1 d\sigma' \Big\{ &(m+1)(n+1)\, p_-^2\ [X(\sigma), X(\sigma')]_{PB} X^m(\sigma) X^n(\sigma') + \\ &+ i(m+1)\, p_-\, X^m(\sigma)\, X^{n+1}(\sigma') - \\ &- i(n+1)\, p_-\, X^n(\sigma')\, X^{m+1}(\sigma) \Big\} \end{aligned} \tag{3.6}$$

where we have used the fact that $X(\sigma)$ and p_- are canonically conjugate. Using the relation eq. (2.23) (understood as a Poisson bracket) and imposing the periodicity of the closed string fields (i.e. $X(1) = X(0)$) we perform one of the integrals and find (ignoring the c-dependent term) that indeed

$$[L_m, L_n]_{PB} = (m-n)\, L_{m+n}. \tag{3.7}$$

For the quantum operators this calculation gets complicated very rapidly with increasing values of m and n as one needs to perform several reorderings that involve evaluating multiple commutators. However, it is possible to organise the calculation in a systematic expansion in powers of $1/p_-$, each order in the expansion being associated with the number of commutators taken. In this expansion one can then check for closure of the $\{L_n\}$ algebra order by order in $1/p_-$. We demonstrate closure on physical states at next to leading order $(p_-)^0$ in appendix C, the classical result being the leading term of order $(p_-)^1$.

Up to this point we have ignored the effect of the c-dependent term in eq. (2.23) in the algebra. The inclusion of this terms modifies the algebra as follows

$$[L_m, L_n] = (m-n)L_{m+n} - \frac{c}{2}(m^2+m)(n+1)\frac{1}{2\pi i}\int_0^1 d\sigma\, X^{m+n-1}(\sigma)\partial_\sigma X(\sigma)\,. \tag{3.8}$$

For $m + n \neq 0$ the second term clearly vanishes as an integral of a total derivative, due to periodicity of the field $X(\sigma)$ on the closed string. However, when $m + n = 0$, the integral reduces to $\int d\ln X$ which, depending on its definition, is either a phase or zero. Note that when $m + n = 0$, the contribution from this term is proportional to $(m^3 - m)$, as expected for a central extension. (Note also that the factor of $1/2\pi i$ that naturally appears in eq. (3.8), as a consequence of the commutation relation eq. (2.23), ensures that this term is a real integer). The definition of this object is related to the definition of the generators $\{L_n\}$ for $n < -1$. However, we can understand its interpretation by following a slightly different route.

Let us consider strings on AdS_3 in global coordinates [7]. Although we cannot directly formulate the worldsheet theory at zero radius in these coordinates, we can do so near the boundary. The boundary of AdS_3 in global coordinates is a cylinder and the asymptotic metric (near the boundary) is

$$\lim_{\rho\to\infty} ds^2 \ \simeq\ R^2(2\, e^{2\rho} d\tilde{x}^+ d\tilde{x}^- + d\rho^2)\,, \tag{3.9}$$

where ρ is the radial coordinate and $\tilde{x}^{\pm} := (\phi \pm \tilde{t})/\sqrt{2}$ with ϕ being the angular variable $0 \leq \phi < 2\pi$ and $\tilde{t}$ being the global time coordinate. With this form of the near boundary metric we can look at the worldsheet theory in light-cone gauge with $R^2/\alpha' = 0$, after redefining $\exp(-\rho) = \tilde{z}$. The worldsheet action, constraints and mode expansions are then similar to those we have already encountered, but with one difference: since ϕ parametrises a circle we can allow for winding modes of the string and the mode expansion of $\tilde{x}^-$ can have a linear term $2\pi w\,\sigma$, with w an integer. Furthermore, in these coordinates we need to adopt a slightly different definition of the spacetime Virasoro generators, so that they naturally generate diffeomorphisms of the circle (at the boundary of AdS_3). This leads us to consider generators in an exponential parametrisation

$$H_n := -\int_0^1 d\sigma\, p_- \, \exp[in\tilde{X}(\sigma)] \,, \qquad \tilde{X}(\sigma) := \tilde{x}^- + \frac{\tilde{z}\dot{\tilde{z}}}{2p_-} \,. \tag{3.10}$$

$\tilde{X}(\sigma)$ (in the asymptotc sense discussed above) obeys the same commutator as in eq. (2.23). Near the boundary, H_n reduce to the Brown-Henneaux Killing vectors in global coordinates. From the commutation relations it then follows that

$$[H_m, H_n] = (m-n)H_{m+n} + \frac{c}{2} m^3\, \delta_{m+n,0} \frac{1}{2\pi} \int_0^1 d\sigma\, \partial_\sigma \tilde{X}(\sigma) \,. \tag{3.11}$$

The presence of a linear term $2\pi w\,\sigma$ in $\tilde{x}^-(\sigma)$ leads to an unambiguous central extension $(c/2) w\, m^3\, \delta_{m+n,0}$. We expect the theory on AdS_3 in global coordinates to be dual to a CFT on the cylinder (or radially quantized CFT), whilst the theory in Poincaré coordinates (discussed previously) should be dual to a CFT on the plane. For the radially quantized theory, the central term above suggests how we should interpret the logarithm in $\int d\ln X$ in eq. (3.8). Noting that, asymptotically, the global coordinate $\tilde{x}^-$ is related to the Poincaré coordinate x^- such that $x^- = \tan\tilde{x}^-$, we must define the logarithm so that it picks out the appropriate phase and the central charges in the two descriptions match.[7] This leads to the following algebra for $\{L_n\}$:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{2} w(m^3 - m)\delta_{m+n,0} \tag{3.12}$$

where w is a 'winding' number defined via the logarithmic integral, as explained above. Thus, from the point of view of the worldsheet theory of the tensionless string on AdS_3, we can see the emergence of a spacetime Virasoro algebra with a potential central extension.[8]

This mechanism indicates that a holographic interpretation as a boundary CFT should be possible for the zero radius (or tensionless) theory (e.g. as would follow from a naive extension of the proposal of [1, 2]). A similar mechanism for generating a spacetime central

[7] The logarithmic integral also yields a phase if we add a small imaginary part to $X(\sigma)$, in our definition of $\{L_n\}$, whose sign depends on how we approach a zero of $X(\sigma)$.

[8] J. Troost has pointed out to us that one can obtain an unambiguous definition of the central term directly in Poincaré coordinates if one imposes an angular identification of the coordinate x in AdS_3. With this identification AdS_3 becomes the vacuum BTZ black hole and the spacetime central charge arises from winding states in this geometry (for a discussion of the tensile string in this geometry see [21]and references therein).

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charge was found in a very different situation in the context of the SL(2, ℝ) WZW model in [14] where the winding of long strings in AdS_3 induced a central charge in the spacetime Virasoro algebra. (A different mechanism by which a spacetime central charge is also induced in the SL(2, ℝ) WZW model was shown to operate in [19].)

3.1 A second copy of the Virasoro algebra

Having argued for the existence of an infinite-dimensional spacetime conformal symmetry (generated by two copies of a Virasoro algebra), we now construct the second copy of the Virasoro algebra. Following a similar approach as before, we first identify two time-independent worldsheet fields

$$Y(\sigma) := x^+(\tau) + \frac{1}{2\pi_+(\sigma)} z\dot{z}(\sigma,\tau)\,, \qquad \pi_+(\sigma) = -\frac{1}{2p_-}\dot{z}^2(\sigma)\,, \tag{3.13}$$

where $x^+(\tau) = p_-\tau$. It is easily verified that the two variables Y and $-\pi_+$ are 'conjugate', in the sense that they satisfy the simple commutation relation,

$$[Y(\sigma),\, -\pi_+(\sigma')] = i\delta(\sigma - \sigma')\,. \tag{3.14}$$

We now propose that the second set of Virasoro generators are

$$\tilde{L}_n := i\int_0^1 d\sigma\, \pi_+(\sigma)\, Y(\sigma)^{n+1} \tag{3.15}$$

which can be shown to commute with each L_n,

$$[L_m, \tilde{L}_n] = 0\,. \tag{3.16}$$

Again, a small z expansion of these conserved quantities yields the following leading terms

$$\tilde{L}_n = i\int_0^1 d\sigma\, \left[\pi_+\,(x^+)^{n+1} + \frac{1}{2}\,(n+1)\,(x^+)^n\, z\,\pi_z - \frac{1}{4}\,n(n+1)\,(x^+)^{n-1}\,z^2\,p_- + \cdots\right] \tag{3.17}$$

which agree with the other set of the asymptotic Killing vectors of AdS_3, namely

$$\begin{aligned}
\tilde{\xi}^+_n &= (x^+)^{n+1} + \mathcal{O}(z^4)\,,\\
\tilde{\xi}^-_n &= -\frac{1}{4}\,n(n+1)(x^+)^{n-1}\,z^2 + \mathcal{O}(z^4)\,,\\
\tilde{\xi}^z_n &= \frac{1}{2}\,(n+1)\,(x^+)^n\,z + \mathcal{O}(z^3)\,,
\end{aligned} \tag{3.18}$$

for all integers n. That $\{\tilde{L}_n\}$ constitute a Virasoro algebra follows from the fact that $-\pi_+$ and Y are canonically conjugate in the sense of eq. (3.14). This is easy to establish classically, at the level of Poisson brackets. For quantum operators two important facts need to be taken into account. First, there is the usual issue of operator ordering. Second, the algebra actually acquires a central extension via the same mechanism encountered in the previous section in the context of the algebra of $\{L_n\}$. To see the emergence of this

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central extension it is convenient to adopt a slightly indirect approach. We first note that the variables X and Y are related by the following relation,

$$p_- X'(\sigma) = \pi_+(\sigma) Y'(\sigma) \,. \tag{3.19}$$

Now the Fourier components of $p_- X'$ (and hence those of $\pi_+ Y'$) are the generators of the worldsheet Virasoro algebra eq. (2.21), so that

$$[\pi_+(\sigma)Y'(\sigma), \pi_+(\sigma')Y'(\sigma')] \;=\; i\pi_+ Y'\left(\frac{\sigma+\sigma'}{2}\right)\delta'\left(\frac{\sigma-\sigma'}{2}\right) - \frac{c}{2}\frac{1}{2\pi i}\,\delta'''(\sigma-\sigma')\,. \tag{3.20}$$

This relation implies that $[Y(\sigma), Y(\sigma')] \sim c\,\delta'(\sigma-\sigma')/\pi_+(\sigma)\pi_+(\sigma')$, leading to a central term in the spacetime Virasoro algebra. Following a short calculation, similar to that in the previous section, we find

$$[\tilde L_m, \tilde L_n] = (m-n)\,\tilde L_{m+n} + \frac{c}{2}\,\tilde w\,(m^3 - m)\,\delta_{m+n,0} \tag{3.21}$$

where $\tilde w = \int_0^1 d\sigma \partial_\sigma \ln Y$ is the 'winding' associated with the worldsheet field $Y(\sigma)$.

L_1, L_0 and L_{-1} and $\tilde L_1$, $\tilde L_0$ and $\tilde L_{-1}$ generate the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ isometry group of AdS_3 and are the conserved charges on the worldsheet corresponding to associated global symmetries. As we discuss in appendix B, this algebra can be shown to close (on physical states) in the quantum theory as well.

3.2 Interpretation via worldsheet symmetries

Although we have argued above that $\{L_n\}$ and $\{\tilde L_n\}$ constitute an infinite-dimensional conformal algebra on zero radius AdS_3 (and demonstrated this by explicit evaluation of commutators), we have not addressed the question of why these conserved quantities realise Virasoro algebras in the first place. Put another way, we know that in general on the worldsheet, the isometry generators of the target space are realised as conserved charges associated with global worldsheet symmetries. (Actually, in light-cone gauge the worldsheet symmetries are combinations of global transformations and compensating worldsheet reparametrisations that preserve the gauge choice.) This is certainly true for the generators of the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ isometry group of AdS_3. It is natural to ask if the same is true for the infinite sets $\{L_n\}$ and $\{\tilde L_n\}$. We now argue that this is indeed the case.

Consider the transformations

$$\delta x^- \;=\; F(X) - \frac{1}{2p_-}\, z\dot z\, F'(X)\,, \tag{3.22a}$$

$$\delta z \;=\; \frac{1}{2}\, z\, F'(X) \tag{3.22b}$$

where F is an arbitrary function. Treating these as "off-shell" transformations, *i.e.* where we neither make use of the equations of motion nor the constraints, we see that they leave the transform the constraint equation (2.6) covariantly. They also transform the second constraint (2.7) covariantly after using the equation of motion. The infinite set of generators of these symmetries, labelled by functions $F_n(X) := X^{n+1}$, indeed constitute a

Virasoro algebra (as can be checked by explicit calculation). The Noether charges for these transformations can then be formally derived by varying the light-cone action eq. (2.5) with the constraints implemented via Lagrange multipliers. Evaluating them on-shell we find that the charges are precisely $\{L_n\}$ (eq. (3.1)) (a given charge L_n follows from the above symmetry with function F_n). The other set of Virasoro generators $\{\tilde{L}_n\}$ correspond to symmetries under the transformations

$$\delta x^- = -\frac{1}{2p_-^2}\,\dot{z}^2\,G(Y) - \frac{1}{2p_-}\,z\dot{z}\,G'(Y) \tag{3.23a}$$

$$\delta z = \frac{1}{2}\,z\,G'(Y) + \frac{1}{p_-}\,\dot{z}\,G(Y)\,, \tag{3.23b}$$

where G is an arbitrary functon. The associated charges are $\{\tilde{L}_n\}$ (eq. (3.15)), with each $\tilde{L}_n$ following from the above symmetry with function $G_n(Y) := Y^{n+1}$.

In a near boundary (or small z) expansion, the transformations above are found to be a combination of Brown-Henneaux spacetime diffeomorphisms and worldsheet reparametrisations that preserve the light-cone gauge. Such a correspondence is to be expected since the Brown-Henneaux transformations (combined with compensating worldsheet reparametrisations) are only asymptotic symmetries of the string action, whilst the transformations above are exact symmetries.

4. Discussions

In this note we have shown that the worldsheet theory of bosonic strings in light-cone gauge on AdS_3 at zero radius possesses symmetries generated by two copies of a Virasoro algebra. We have shown that the conserved charges on the worldsheet, associated to these symmetries, can naturally be interpreted as generators of a spacetime Virasoro algebra. This interpretation is prompted primarily by the observation that, near the boundary of AdS_3, the conserved charges are isomorphic to the Brown-Henneaux diffeomorphisms that generate asymptotic isometries of AdS_3. We also find, from the worldsheet viewpoint, that this spacetime algebra can pick up a central extension indicating a potential holographic interpretation as a dual conformal field theory in two dimensions. This is consistent with the proposal of [1, 2], that bosonic string theory on zero radius AdS_{d+1} should be dual to free scalar field theory in d dimensions. The appearance of the spacetime Virasoro algebra is intimately related to a *worldsheet* Virasoro algebra satisfied by an infinite set of conserved quantities on the worldsheet of the tensionless string. The central extension in spacetime which is generated by the winding of a worldsheet field is also directly related to the central charge in the worldsheet algebra. These features are similar to those encountered in a different physical context: that of strings on AdS_3 with an NS-NS two form gauge field, which can be described as an SL$(2,\mathbb{R})$ WZW model.

Strings on $AdS_3 \times S^3 \times M^4$: as mentioned in the introduction, we can also look at the tensionless limit (or zero AdS radius limit) of IIB string theory on $AdS_3 \times S^3 \times M^4$ (M^4 being T^4 or K^3) with R-R three-form flux. For any finite radius, this theory (which describes the near horizon limit of the D1-D5 brane system) is dual to a $1+1$-dimensional sigma model whose target space is a deformation (blowup) of the symmetric product space $(M^4)^k/S_k$ with $k = Q_1 Q_5$ for Q_1 D1-branes and Q_5 D5-branes. It is likely that the limit of zero radius describes the CFT of the symmetric product (at which point a supergravity description is not valid). It would be extremely interesting to understand how this actually works, given that a light-cone formulation of the superstring on this background (for any radius) already exists [8, 9]. The analysis of our paper pertains only to the bosonic sector of this theory in the tensionless limit.[9]

Virasoro algebras for AdS_d: Although we have restricted our discussion to AdS_3, it turns out that the conserved charges $\{L_n\}$ and $\{\tilde{L}_n\}$ and the Virasoro symmetries of the tensionless string theory continue to exist for AdS space of any dimension. In particular, we can simply extend our definitions of the time-independent fields X and Y for the theory on AdS_d as

$$X := x^- + \frac{1}{2p_-} z^i \dot{z}^i \,, \qquad Y := x^+ + \frac{1}{2\pi_+} z^i \dot{z}^i \tag{4.1}$$

with $i = 1, \ldots, d-2$. The charges L_n and $\tilde{L}_n$, defined as in eqs. (3.1) and (3.15), then still satisfy a Virasoro algebra. The appearance of these symmetries for AdS_d suggests a relation to the generalisation to higher dimensions of the three-dimensional Brown-Henneaux symmetries, discovered in [22].[10][11] It would be interesting to explore this connection further (We show how this could be made precise in appendix D.). Another interesting question is whether these symmetries can be extended to the tensile (super-)string on higher-dimensional AdS spaces (with $R^2/\alpha' \neq 0$).

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[9]It is worth noting that a naive analysis, based on the Brown-Henneaux formula for the central charge in the Virasoro algebra: $c = 3R/2G_N$ (where G_N is the three-dimensional Newton constant), would suggest that zero AdS_3 radius implies a zero central charge. This naive expectation clashes with an expectation based on the duality proposal of [1, 2] or the singular limit of the D1-D5 system. However, we are discussing a limit $R^2/\alpha' \to 0$ wherein it should be possible to keep R/G_N fixed, as suggested by the $D1-D5$ example. It is also conceivable that the Brown-Henneaux formula, which is valid at large radius, could receive corrections in the opposite limit, i.e. the small radius limit.

[10]We would like to thank A. Karch for pointing out the possible connection to [22, 23].

[11]Geometrically, the reason that these asymptotic Brown-Henneaux type isometries of AdS_d exist is that this geometry admits a foliation with codimension $d-3$ AdS_3 slices [23]. Thus these generalised transformations are just the ordinary Brown-Henneaux diffeomorphisms acting on a given AdS_3 slice (for fixed values of the $d-3$ transverse coordinates). We thank A. Chamblin for explaining this to us.

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A. Operator ordering

In this appendix we review the issue of defining operators (for example P_σ) in the quantum theory which involve two or more powers of the modes z_n and p_n, and are therefore subject to ordering ambiguities. In the context of tensionless strings in flat space (null-strings), this has been analysed in some detail in [17] where they focussed on two possible definitions of quantum operators — referred to as Weyl (W) and 'normal' ordered (N). Usually, Weyl (or hermitian) ordering is taken to mean the symmetrization of position and momentum operators in a given expression so that the resulting operator is hermitian. In the present context this ordering turns out to be equivalent to placing all momentum modes p_n to the right of all position modes z_n. On the other hand, 'normal' ordering is a prescription where all positive frequency modes $z_{n>0}$ and $p_{n\geq 0}$ are ordered to the right of all negative frequency modes. In what follows we focus on some of the main consequences of adopting either approach in the context of worldsheet algebras. We will not atempt to discuss the spectrum of physical states of the spacetime theory.

The light-cone string hamiltonian H_{LC} has no ordering ambiguities and in terms of Fourier modes is

$$H_{LC} = \frac{1}{2}\sum_{n=-\infty}^{\infty} p_{-n}p_n\,. \tag{A.1}$$

For the two types of orderings, we define the vacuum $|0\rangle$ of the light-cone hamiltonian via

$$p_n|0\rangle_W = 0\,, \tag{A.2}$$

(for all n) for Weyl ordering (W) and

$$p_{n\geq 0}|0\rangle_N = 0\,, \qquad z_{n>0}|0\rangle_N = 0\,, \tag{A.3}$$

for normal ordering (N). Both definitions are consistent with canonical commutation relations although the Weyl ordered definition is perhaps intuitively obvious.

The worldsheet spatial momentum P_σ (the coefficient of the linear term in eq. (2.11)) is defined as

$$P_\sigma^{(W)} = \frac{2\pi i}{p_-}\sum_{n=-\infty}^{\infty} n\, z_{-n}p_n\,, \qquad P_\sigma^{(N)} = \frac{2\pi i}{p_-}\sum_{n=1}^{\infty}\left(z_{-n}p_n - p_{-n}z_n\right), \tag{A.4}$$

in the two different schemes. Both definitions annihilate the respective vacuum state. Finally, for physical states of the closed string we must impose

$$P_\sigma^{(N),(W)}|\text{phys}\rangle_{N,W} = 0 \tag{A.5}$$

in both quantisation schemes which enforces the periodicity of the worldsheet field $x^-(\sigma,\tau)$.

Given these definitions of the Weyl ordered and normal ordered vacua, we can derive the central terms in the worldsheet Virasoro algebras eqs. (2.17) and (2.21) by following the standard trick of evaluating the commutators on the respective vacuum states, $\langle 0|[l_n, l_{-n}]|0\rangle$. The coefficients c and c' in the central extension can be determined by evaluating $\langle 0|l_1 l_{-1}|0\rangle$ and $\langle 0|l_2 l_{-2}|0\rangle$ in two different ways (directly, and by making use of the

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Virasoro algebra itself). We find that for the Weyl ordered approach $c = c' = 0$ whilst for the normal ordered approach $c = 2$ and taking into account the ordering constant (equal to $-1/12$) in l_0 we find that $c' = 0$.

B. AdS_3 isometry algebra

The isometry group of AdS_3 is generated by $\{L_1, L_0, L_{-1}\}$ and $\{\tilde{L}_1, \tilde{L}_0, \tilde{L}_{-1}\}$ whose (classical) expressions can be found in eqs. (3.1) and (3.15). Here we give explicit expressions for these generators at the quantum level, in terms of which their algebra closes. There are no subtleties involved in the Weyl ordered approach, the classical calculation and the quantum calculation are identical and one does not have to perform any additional reorderings of expressions after the evaluation of the commutators (note that this applies only to the $sl(2,\mathbb{R}) \times sl(2,\mathbb{R})$ algebra, not to the entire Virasoro algebra). For normal ordering, however the situation is different. The normal ordered expressions for the generators are

$$\begin{aligned}
L_1 &= \frac{i}{p_-}\sum_{n\neq 0}\sum_{m,m'}\left(\frac{m}{n}+\frac{1}{2}\right)\left(\frac{m'}{n}-\frac{1}{2}\right) : z_{-n-m}p_m z_{n-m'}p_{m'} : -\frac{i}{3}p_- P_\sigma^2 \\
&\quad -ip_-(x_0^-)^2 - ip_- x_0^- \frac{1}{2p_-}\sum_m : z_{-m}p_m : \\
&\quad -\frac{i}{2}x_0^- \sum_m : z_{-m}p_m : -\frac{i}{4p_-}\sum_{m,m'} : z_{-m}p_m z_{-m'}p_{m'} : - \\
&\quad -\frac{i}{2}p_- P_\sigma x_0^- - \frac{i}{2}\sum_m : z_{-m}p_m : P_\sigma - \frac{i}{2}p_- x_0^- P_\sigma - \\
&\quad -i\sum_{n\neq 0}\sum_{m,m'}\frac{1}{p_-}\frac{m}{n}\left(\frac{m'}{n}-\frac{1}{2}\right) : z_{-m}p_m z_{n-m'}p_{m'} : - \\
&\quad -i\sum_{n\neq 0}\sum_{m,m'}\frac{1}{p_-}\frac{m}{n}\left(\frac{m'}{n}-\frac{1}{2}\right) : z_{n-m'}p_{m'} z_{-m}p_m : \qquad \text{(B.1a)}
\end{aligned}$$

$$L_0 = -\frac{ip_- x_0^- ip_- P_\sigma}{2} - \frac{i}{2}\sum : z_{-m}p_m : \qquad \text{(B.1b)}$$

$$L_{-1} = -ip_- \qquad \text{(B.1c)}$$

$$\tilde{L}_1 = -i\frac{p_-}{2}\sum z_{-m}z_m \qquad \text{(B.1d)}$$

$$\tilde{L}_0 = \frac{i}{2}\sum : z_{-m}p_m : + \text{const} \qquad \text{(B.1e)}$$

$$\tilde{L}_{-1} = \frac{-i}{2p_-}\sum p_{-m}p_m \,. \qquad \text{(B.1f)}$$

In the above expressions, normal ordering is denoted :: and implies that all the positive frequency modes of $\{z_n\}$ and $\{p_n\}$ are put to the right so that they annihilate the vacuum. We also choose to keep all powers of P_σ to the right so that they annihilate all physical states. It is fairly easy to see that both $sl(2,\mathbb{R})$ algebras close individually. However, it requires a long and tedious calculation to establish that the two $sl(2,\mathbb{R})$ algebras commute with each other — in particular that L_1 commutes with $\{\tilde{L}_1, \tilde{L}_0, \tilde{L}_{-1}\}$. One useful property that simplifies the calculation somewhat is that P_σ commutes with $\{\tilde{L}_1, \tilde{L}_0, \tilde{L}_{-1}\}$.

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C. Quantum closure of the Virasoro algebra

In this appendix we demonstrate that the set of conserved charges $\{L_n\}$ form a Virasoro algebra on physical states. Instead of discussing the calculation in its entirety, for clarity, we will just demonstrate the closure of the commutator algebra at next to leading order (beyond the leading order classical Poisson bracket calculation). In this context, "order" means the number of individual commutators that must be evaluated in reordering the result of a given term in the general $[L_m, L_n]$ commutator[12] Keeping track of the order of terms in the calculation is made easier by the fact that both the commutators $[X(\sigma), X(\sigma')]$ and $[X(\sigma), p_-]$ reduce the power of p_- in a given term by one. Since each L_n has just one power of p_- then this means that we can express $[L_m, L_n] = \sum_{k=1}^{\infty} (p_-)^{2-k} \mathcal{O}_k$ for some set of p_--independent operators $\{\mathcal{O}_k\}$. Closure of the algebra then requires that both $p_- \mathcal{O}_1 = (m-n)L_{m+n}$ and $\mathcal{O}_k = 0$ for all $k > 1$ (up to constraint terms that annihilate physical states).

In order to verify this statement to order $k = 2$, we must first calculate the $[X(\sigma), L_n]$ commutator, keeping only leading order $((p_-)^0)$ and next to leading order $((p_-)^{-1})$ terms. The result is that

$$
\begin{aligned}
[X(\sigma), L_n] = & \left(X^{n+1}(\sigma) + \frac{1}{2}(X^{n+1}(1) - X^{n+1}(0))\right) \\
& +(n+1)\, X'(\sigma) \int_0^1 d\sigma' \, (h(\sigma'-\sigma) - \sigma')\, X^n(\sigma') + \\
& + \frac{n(n+1)}{2} \frac{i}{p_-} \Bigg\{ X'(\sigma) \int_0^1 d\sigma' \, (h(\sigma'-\sigma) - \sigma')\, X^{n-1}(\sigma') + \\
& \qquad + \frac{1}{n}\left(X^n(\sigma) + \frac{1}{2}(X^n(1) - X^n(0)) - \int_0^1 d\sigma' \, X^n(\sigma') \right) - \\
& \qquad - \int_0^1 d\sigma' \, (h(\sigma'-\sigma) - \sigma')\, X^{n-1}(\sigma') \times \\
& \qquad \times \Big\{ (h(\sigma'-\sigma) - \sigma')\, X''(\sigma) + (\delta(\sigma-\sigma') - 1)\, X'(\sigma) + \\
& \qquad\quad + \delta(\sigma - \sigma') X'(\sigma') \Big\} \Bigg\}
\end{aligned} \tag{C.1}
$$

up to terms involving $X(1) - X(0) = P_\sigma$ ordered to the right (which annihilates physical states) plus further terms of order $(p_-)^{-2}$. The function h is defined by $h(\sigma) := \sigma - 1/2\,\mathrm{sgn}(\sigma)$, where $\mathrm{sgn}(\sigma)$ equals plus or minus one when σ is respectively positive or negative. The above result then implies that

$$
[L_m, L_n] = (m-n)\, L_{m+n} \tag{C.2}
$$

up to terms involving $X^r(1) - X^r(0)$ (for various powers r) which we take to annihilate physical states, plus further terms of order $(p_-)^{-1}$. (Note that classically, periodicity on

[12] For example, in the Poisson bracket calculation all the terms in each L_n are just treated as fields (rather than operators) and can therefore be ordered arbitrarily. Hence, after evaluating the first Poisson bracket of all such terms with each other, the resulting expressions can be reordered at the expense of no further Poisson brackets.

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the closed string implies $X^r(1) - X^r(0) = 0$, but quantum mechanically we can only require that such operators annihilate physical states of the theory.) In particular this shows that $p_- \mathcal{O}_1 = (m-n)L_{m+n}$ and $\mathcal{O}_2 = 0$ as claimed.

D. Virasoro algebras for AdS_d

Although the main topic of this note is AdS_3, it turns out that straightforward generalisations of the conserved charges $\{L_n\}$ and $\{\tilde{L}_n\}$ also exist in AdS space of any dimension d. We identify two possible generalisations to AdS_d. The first is obtained by defining a time-independent field $X(\sigma)$ as before,

$$X := \left(x^- + \frac{1}{2p_-} z^i \dot{z}^i \right) \tag{D.1}$$

where $i = 1, \ldots, d-2$ run over the directions transverse to the light-cone on AdS_d in Poincaré coordinates. It is a simple excercise to demonstrate that, with this generalised X, $\{L_n\}$ (as defined in eq. (3.1)) continue to satisfy a Virasoro algebra. The same is true for $\{\tilde{L}_n\}$ (using the obvious generalisation of Y). The existence of these Virasoro algebras for general d is somewhat mysterious. It has been noted in [22] that the Brown-Henneaux symmetries can be generalised to general AdS_d spacetimes. However, the generalised $\{L_n\}$ and $\{\tilde{L}_n\}$ above do not match with the generalised Brown-Henneaux diffeomorphisms in [22]. Hence the geometrical interpretation of our generalised Virasoro algebra is not entirely clear. Despite this, we are able to identify another set of conserved quantities on the string worldsheet which do reduce to the diffeomorphisms in [22], near the AdS_d boundary.

To construct these conserved quantities, we begin by defining the following matrix of time-independent worldsheet fields

$$X^{ij} := \delta^{ij} \left(x^- + \frac{1}{2p_-} z^k \dot{z}^k \right) + \frac{i}{2p_-} \sqrt{\frac{(d-2)}{2}} \left(z^i \dot{z}^j - z^j \dot{z}^i \right). \tag{D.2}$$

Notice that X^{ij} correctly reduces to X (defined in eq. (2.19)) for $d = 3$. The corresponding generalisation of Y is given by the matrix of conserved quantities Y^{ij}, defined just as X^{ij} but with x^- and p_- replaced by x^+ and π_+ respectively (this definition reduces to eq. (3.13) for $d = 3$). We now define the conserved charges

$$\mathcal{L}_n := -\frac{i}{d-2} \int_0^1 d\sigma \, p_- \, \mathrm{tr} \left(X^{n+1} \right) (\sigma), \tag{D.3}$$

where powers of matrices X^{ij} are taken with respect to the usual matrix product and the trace is taken by contracting δ^{ij} with the remaining two free indices. This definition correctly reduces to the definition of L_n in eq. (3.1) for $d = 3$. The corresponding expressions for the conserved charges $\tilde{\mathcal{L}}_n$ are defined just as $\mathcal{L}_n$, but with X^{ij} and p_- replaced with Y^{ij} and $-\pi_+$ respectively. This also correctly reduces to the definition of $\tilde{L}_n$ in eq. (3.15) for $d = 3$.

Performing a small z expansion of $\mathcal{L}_n$ gives the following leading order terms

$$\mathcal{L}_n = -i \int_0^1 d\sigma \left[p_- (x^-)^{n+1} + \frac{1}{2}(n+1)(x^-)^n \, z^i \, \pi_z^i - \frac{1}{4} n(n+1)(x^-)^{n-1} \, z^i z^i \, \pi_+ + \cdots \right], \tag{D.4}$$

where we have used the fact that p_-, $\pi_+ = \dot{x}^- = -\dot{z}^i \dot{z}^i / 2p_-$ and $\pi_z^i = \dot{z}^i$ are the canonical momenta conjugate to the coordinates x^-, x^+ and z^i respectively. Similarly, an expansion of $\tilde{\mathcal{L}}_n$ near the AdS_d boundary yields

$$\tilde{\mathcal{L}}_n = i \int_0^1 d\sigma \left[\pi_+ (x^+)^{n+1} + \frac{1}{2}(n+1)(x^+)^n \, z^i \, \pi_z^i - \frac{1}{4} n(n+1)(x^+)^{n-1} \, z^i z^i \, p_- + \cdots \right]. \tag{D.5}$$

In Poincaré coordinates, these precisely correspond to the asymptotic isometry generators of AdS_d found in [22].

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